LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

1ST SEMESTER 2022/2023 ACADEMIC YEAR

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (STATISTICS)

STAT 415: STOCHASTIC PROCESSES

STREAM: R ***TIME: 2 HRS***

DAY: WED [8.30AM – 10.30AM] ***DATE: 14/12/2022***

THIS QUESTION PAPER CONSISTS OF FOUR (4) PAGES

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INSTRUCTIONS: Answer **QUESTION ONE** and any other **TWO** questions

QUESTION ONE (30 MARKS)

a) Briefly explain the meaning of the following terms as used in stochastic processes:

   i) State space

   ii) Markov process

   iii) Irreducible Markov chain

(**6 Marks**)

b) A Markov chain $\{X_n\}_{n\geq 0}$ on the states 0, 1, 2 has the transition probability matrix $P = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$ and initial distribution $p_0 = \Pr(X_0 = 0) = 0.3, p_1 = \Pr(X_0 = 1) = 0.4$ and $p_2 = \Pr(X_0 = 2) = 0.3$.

Compute

i) $P(X_0 = 1, X_1 = 2, X_2 = 1)$

ii) $P(X_2 = 1,\ X_3 = 2/X_1 = 0)$

iii) $P(X_4 = 2/X_1 = 0)$

(**8 Marks**)

c) Consider a Markov chain consisting of six states $\{0, 1, 2, 3, 4, 5\}$ and having the following transition probability matrix

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

i) Draw the state transition diagram for this chain

ii) Determine the classes and specify which are recurrent and which are transient

(**6 Marks**)

d) Customers arrive at a service facility according to a Poisson process of rate $\lambda = 6$ per hour. Given that 10 customers arrived during the first four hours of service, what is the conditional probability that 5 customers arrived during the first hour?

(**4 Marks**)

e) Let $\{X_0 = 1, X_1, X_2, \cdots\}$ be a branching process with offspring distribution given by $p_0 = \frac{1}{10}, p_1 = \frac{2}{5}$ and $p_2 = \frac{1}{2}$. Find the mean and variance of X_5, the population at generation 5.

(**6 Marks**)

QUESTION TWO (20 MARKS)

Let $\{X_n\}_{n\geq 0}$ be a Markov chain on states $\{1, 2, 3,4\}$ and having transition probability matrix

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a) Draw the transition diagram for this chain

(3 Marks)

b) Calculate f_{34} the probability of absorption into state 4 starting from state 3.

(5 Marks)

c) Compute f_{ii} for $= 1, 3, 4$, the probability that starting from state i, the chain will ever reenter i.

(8 Marks)

d) Use the values of f_{ii} in part c) to classify each of the four states as either transient or persistent

(4 Marks)

QUESTION THREE (20 MARKS)

a) Briefly explain the difference between the transient and recurrent states in a Markov chain.

(4 Marks)

b) Let $\{X_n: n \in \mathbb{N}\}$ be a Markov chain with state space S. Prove that if a $j \in S$ communicates with a recurrent state $i \in S$, then j is also recurrent.

(6 Marks)

c) An auto insurance company classifies its customers in three categories: poor, satisfactory and preferred. No one moves from poor to preferred or from preferred to poor in one year. 40% of the customers in the poor category become satisfactory, 30% of those in the satisfactory category moves to preferred, while 10% become poor; 20% of those in the preferred category are downgraded to satisfactory.

i) Write the transition probability matrix for this model

ii) Draw the transition diagram for the Markov chain

iii) Find the stationary distribution π of the Markov chain

(10 Marks)

QUESTION FOUR (20 MARKS)

a) Let $\{X(t): t \geq 0\}$ be a Poisson process having a rate parameter $\lambda = 4$. Determine the following expectations:

i) $E\left[\left\{X(2)\right\}^2\right]$

ii) $E[X(1)X(5)]$

(6 Marks)

b) The number of customers arriving at a grocery store can be modeled by a Poisson process of rate $\lambda = 4$ customers per hour. Given that the store opens at 10 AM, what is the probability that two customers have arrived by 10.20 AM and a total of six have arrived by 11.30 AM?

(6 Marks)

c) Let $\{X(t): t \geq 0\}$ be a Poisson process of rate λ. Find the conditional probability that there were m events in the first t hours, given that there were n events in the first T hours. Assume $0 \leq m \leq n$ and $0 < t < T$.

(8 Marks)

QUESTION FIVE (20 MARKS)

a) Let $\{X_0 = 1, X_1, X_2, \cdots\}$ be a branching process with offspring distribution given by $p_0 = \frac{1}{4}, p_1 = \frac{1}{4}$ and $p_2 = \frac{1}{2}$. Find the probability that the process will eventually die out.

(5 Marks)

b) Let $X_1, X_2, \ldots,$ be a sequence of independent and identically distributed random variables with mean μ and variance σ^2. Let N be a non-negative integer valued random variable independent of the sequence $X_i, i \geq 1$. Show that $Var(\sum_{i=1}^{N} X_i) = \sigma^2 E(N) + \mu^2 Var(N)$.

(6 Marks)

c) Let $\{X_n\}_{n\geq 0}$ be a branching process with $X_0 = 1$. Let μ and σ^2 be the mean and variance, respectively, of the offspring distribution. Prove that $Var(X_n) = \begin{cases} \sigma^2 n & if\ \mu = 1 \\ \sigma^2 \mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right) & if\ \mu \neq 1 \end{cases}$.

(9 Marks)