LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

SECOND SEMESTER 2023/2024 ACADEMIC YEAR

THIRD YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (GENERAL)

MATH 313: REAL ANALYSIS II

STREAM: R ***TIME: 2 HRS***

DAY: MONDAY [11.30A.M – 1.30P.M ***DATE: 08/04/2024***

THIS QUESTION PAPER CONSISTS OF FOUR (4 PAGES

PLEASE DO NOT OPEN UNTIL THE INVIGILATOR SAYS SO.

Answer question **ONE** and any other **TWO** questions.

QUESTION ONE (30 MARKS)

a. With respect to a function $f:[a,b]\to\Re$, define the terms

i) Bounded variation **(2 marks)**

ii) Total variation **(2 marks)**

b. Evaluate the following R-S integrals.

i) $\int_0^{\pi/2} \sin^2 x\, d(\sin x + x)$. **(3 marks)**

ii) $\int_0^{5} x^2 d\left(|x| - [x]\right)$. **(4 marks)**

c. Suppose that f is continuous on $[a,b]$ and differentiable on (a,b) . if $f'(x)\geq 0 \quad \forall x\in(a,b)$,show that f is monotone increasing. (4 marks)

d. Let $f:\Re\to\Re$ be defined by $f(x)=6x-3$ on $[3,6]$. By dividing $[3,6]$ into n equal subintervals, show that f is Riemann integrable. (5 marks)

e. Show that $F(x)=x\ln x - x$ is an antiderivative of $f(x)=\ln x$ on $(0,\infty)$ and hence deduce that $\int_0^1 \ln(1+x)\,dx = \ln\left(\frac{4}{e}\right)$. **(4 marks)**

f. Evaluate $\Gamma(2)$. **(3 marks)**

g. Use ratio test to show that the series $\sum_{n=1}^{\infty}\frac{x^n}{n!}$ converges for all values of x. **(3 marks)**

QUESTION TWO (20 MARKS)

a. Let $\{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$ and let f be a decreasing function on $[a,b]$. Show that the sum of the jumps at these points is always bounded by $f(a)-f(b)$. **(5 marks)**

b. Suppose f is continuous on $[a,b]$ and f' exist and $|f'(x)|\leq A \ \forall x\in(a,b)$, show that $f\in BV[a,b]$. (where BV is a bounded variation) **(6 marks)**

c. Let $P=\{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$. Define the terms upper Darboux sums, $U(f,P)$, and lower Darboux sums, $L(f,P)$. (4 marks)

d. Show that the function f defined on the interval $[a,b]$ by $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$ is not Riemann integrable. **(5 marks)**

QUESTION THREE (20 MARKS)

a. If P^* is a refinement of P, show that $L(f,P,\alpha) \le L(f,P^*,\alpha)$. **(6 marks)**

b. Distinguish between Riemann integral and Riemann- Stieltjes integral. **(4 marks)**

c. Show that if f is a constant on the interval $[a,b]$, then it is R-S integrable with respect to any function α and

$$\int_a^b f \; d\alpha = f(a)[\alpha(b) - \alpha(a)].$$ **(6 marks)**

d. Prove that if f is monotonic increasing on $[a,b]$, then f is of bounded variation.

(4 marks)

QUESTION FOUR (20 MARKS)

a. State and prove the necessary and sufficient condition for the convergence of a series. .

. **(7 marks)**

b. Differentiate between absolute convergence and conditional convergence. **(3 marks)**

c. Prove that if $\sum a_n$ is absolutely convergent, then it is convergent. **(5 marks)**

d. Determine the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$. **(5 marks)**

QUESTION FIVE (20 MARKS)

a. Suppose f and g are continuous on $[a,b]$, prove the Cauchy –Schwartz inequality

$$\left[\int_a^b f(x)g(x)\,dx\right]^2 \le \int_a^b [f(x)]^2\,dx . \int_a^b [g(x)]^2\,dx.$$ **(7 marks)**

b. show that $\int_{\underline{a}}^{b} f\,d\alpha \le \int_a^{\bar{b}} f\,d\alpha$ **(7 marks)**

c. Evaluate the R-S integrals

i) $\int_0^4 (3x-1)\,d\,x^2$ **(3 marks)**

ii) $\int_0^1 e^{x^2}\,d(2x^2+1)$ **(3 marks)**