LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

1ST SEMESTER 2022/2023 ACADEMIC YEAR

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (GENERAL)

MATH 414: MATHEMATICAL MODELLING

STREAM: R ***TIME: 2 HRS***

DAY: WED [8.30AM – 10.30AM] ***DATE: 21/12/2022***

THIS QUESTION PAPER CONSISTS OF FOUR (4) PAGES

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INSTRUCTIONS TO CANDIDATES

- Answer Question ONE (COMPULSORY) and ANY TWO Questions
- Do not write on the question paper.
- LAIKIPIA UNIVERSITY observes ZERO tolerance to examination cheating.

QUESTION ONE (30 MARKS)

(a). What is mathematical modelling (2 marks)

(b). State any three reasons why mathematical modelling is important (3 marks)

(c). Consider the single species model given by $\frac{dN}{dt} = rN, N(0) = N_0, r = (\lambda - \mu)$

(i). State any two assumptions (2 marks)

(ii). Solve and explain the solution of the model (5 marks)

(d). Suppose N_1 and N_2 are interacting species. Let N_1 be the population density of the prey and N_2 be the population density of the predator. Define a_1 and a_2 to be per capita birth rate and death rate of prey and predator respectively, and b_{12} and b_{21} be the rate of predation. Derive a prey-predator model (6 marks)

(e). Consider the logistic model below $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), N(0) = N_0$

(i). Find the equilibrium points and determine their stability (5 marks)

(ii). Solve the logistic model above (5 marks)

(iii). Sketch the phase portrait for the logistic model above (2 marks)

QUESTION TWO (20 MARKS)

(a) . The initial population of a certain bacteria was 300, after 24 hours, the population was found to be 500. Assuming a Malthusian model.

(i) What is the reproductive growth rate (3 marks)

(ii) What will be the number of bacteria after 5 days (2 marks)

(b). Let x and y be the population of two competing species
$$\frac{dt}{dt} = 10x - 5xy$$
$$\frac{dy}{dt} = 3y + xy - 3y^2$$

(i) define clearly with reasons the kind of interaction represented by the above model. (3 marks)

(ii) Determine all its equilibrium points (4 marks)

(iii) Determine the type of stability of all the equilibrium points in (ii) above (8 marks)

QUESTION THREE (20 MARKS)

(a). State and explain the six main stages involved in mathematical modelling (12 marks)

(b). State and explain any three characteristics of a population (3 marks)

(c) . Suppose you start a saving account with a Ksh. 5000 initial savings and decide to save Ksh. 2000 each year. Assuming that your savings will earn 5% per annum, compound interest, write a model and determine how much you will have accumulated after 25 years of saving. (5 marks)

QUESTION FOUR (20 MARKS)

(a). In brief explain any two limitations of mathematical modeling (2 marks)

(b). Consider the epidemic model with constant recruitment into the susceptible population

$$\frac{dS}{dt} = \Lambda - \beta S\frac{I}{N} - \mu S$$
$$\frac{dI}{dt} = \beta S\frac{I}{N} - (\gamma + \mu)I \text{ , where } N(t) = S(t) + I(t) + R(t)$$
$$\frac{dR}{dt} = \gamma I - \mu R$$

(i). Determine the limiting value of the model solutions (5 marks)

(ii). Determine the disease free equilibrium point (3 marks)

(iii). Linearize the above model at the disease free equilibrium point (4 marks) (iv). Find the eigenvalues of the linearization matrix above (3 marks)

(v). Study the stability of the disease free equilibrium point and deduce the basic reproductive number of the model (3 marks)

QUESTION FIVE (20 MARKS)

(a). A confectioner manufactures two kinds of candy bars: "Proteinplus", that has no carbohydrates, and "Sugar plus", with no fat. Proteinplus sells for a profit of 40 cents per bar and Sugarplus sells for a profit of 50 cents per bar. The candy is processed in three main operations: blending, cooking and packaging. The following table records the average time in minutes required by each bar for each of the processing operations.

	Blending	Cooking	Packaging
ProteinPlus	1	5	3
SugarPlus	2	4	1

During each production run the blending equipment is available for maximum of 12 machine hours, the cooking equipment is available for at most 30 machine hours, and the packaging equipment for no more than 15 hours. If the machine time can be allocated to the making of either candy type at all times that is available, the confectioner wants to know how many boxes of each type should be produced in order to realize the maximum profit.

(i) Formulate the problem as a linear program (4 marks)

(ii) Sketch the feasible region and optimal isoprofit line (6 marks)

(iii) Find the optimal solution. (2 marks)

(b). Alicia opens an account that pays an annual rate of 6% compounded continuously with an annual investment of ksh.5000. After that she deposits an additional ksh.1200 per year. Assuming that no withdrawals are made and continues with the same yearly deposit over a period of 10 years, how much will be in the account at the end of the ten-year period. (8 marks)