LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

1ST SEMESTER 2022/2023 ACADEMIC YEAR

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (STATISTICS)

STAT 412: INTRODUCTION TO MEASURE AND PROBABILITY

STREAM: R ***TIME: 2 HRS***

DAY: WED [8.30AM – 10.30AM] ***DATE: 21/12/2022***

THIS QUESTION PAPER CONSISTS OF FOUR (4) PAGES

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INSTRUCTIONS: Answer question ONE and any other TWO questions

QUESTION ONE (30 MARKS)

a) Let Ω be a set, and let $\mathcal{F} \subset \mathcal{P}(\Omega)$ be some set of subsets of Ω. Assume that $\Omega \in \mathcal{F}$ and that $A\backslash B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$. Show that $\mathcal{F}$ is an algebra.

(3 Marks)

b) Let μ^* be the Lebesgue outer measure on $\mathbb{R}$. Prove that $A \cup B$ is Lebesgue measurable whenever A and B are.

(5 Marks)

c) For a nonnegative integer valued random variable N, show that $\sum_{i=0}^{\infty} i\mathbb{P}(N > i) = \frac{1}{2}[E(N^2) - E(N)]$

(5 Marks)

d) Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $B \epsilon \mathfrak{N}$ satisfies $\mathbb{P}(B) > 0$. Show that$\mathbb{Q}: \mathcal{F} \rightarrow [0,1]$, where $\mathbb{Q}(A) = \mathbb{P}(A/B)\ \forall A \in \mathcal{F}$, is a probability measure on Ω. If $C \in \mathcal{F}$ and $\mathbb{Q}(C) > 0$, show that $\mathbb{Q}(A/C) = \mathbb{P}(A/B \cap C)$.

(7 Marks)

e) Let $X_1, X_2, \dots, X_n$ be independent $N(0,1)$ random variables. Find the characteristic function of the random variable $Y = \sum_{i=1}^{n} X_j^2$and identify its distribution.

(6 Marks)

f) Let X be a random variable with density function $f(x) = \begin{cases} \frac{5}{x^6}, & x \geq 1 \\ 0 & elsewhere \end{cases}$ for ≥ 1 . Give a bound using Chebyshev's inequality for $\mathbb{P}(X < 2)$.

(5 Marks)

QUESTION TWO (20 MARKS)

a) Let $\mathcal{F}$ is a σ – algebra of subsets of Ω. Prove that

i) if $A_1,\ A_2, \dots, A_n \in \mathcal{F}$, then $A_1 \cap A_2 \cap \dots \cap A_n \in \mathcal{F}$.

ii) If $A, B \in \mathcal{F}$, then $A \triangle B \in \mathcal{F}$

(6 Marks)

b) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $E \in \mathcal{F}$. Define $\mu_E: \mathcal{F} \longmapsto [0, +\infty]$ by $\mu_E(A) = \mu(A \cap E),\ A \in \mathcal{F}$. Prove that μ_E is a measure.

(6 Marks)

c) Let μ be a measure on an algebra $\mathscr{F}$, and let $A_1, A_2, \dots, A_n \in \mathcal{F}$, $1 \leq n < \infty$. Prove that $\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i)$.

(8 Marks)

QUESTION THREE (20 MARKS)

a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Prove that $|\mathbb{P}(A) - \mathbb{P}(B)| \leq \mathbb{P}(A\backslash B) + \mathbb{P}(B\backslash A)$, for all $A, B \in \mathcal{F}$.

(3 Marks)

b) Let $A_1, A_2, \ldots, A_n$ be events in the sample space Ω. If $A_1, A_2, \ldots, A_n$ are independent and $\mathbb{P}(A_i) = \theta, i = 1,2, \ldots, n$, find an expression for $\mathbb{P}(\cup_{i=1}^{n} A_i)$

(4 Marks)

c) A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 85% of the time if the person has the virus and 5% of the time if the person does not have the virus. Find the probability that a person does not have the virus given that he has tested negative

(8 Marks)

d) Let X be a binomial random variable with parameters n and p. Show that

$$E\left[\frac{1}{X+1}\right] = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

(5 Marks)

QUESTION FOUR (15 MARKS)

a) Let Z be a standard normal random variable Z. Show that $E[Z^{n+1}] = nE[Z^{n-1}]$. Use the result to compute $E[Z^4]$

(6 Marks)

b) A prisoner is trapped in a cell containing 3 doors. The first door leads to a tunnel that returns him to his cell after 1 day of travel. The second leads to a tunnel that returns him to his cell after 4 days' travel. The third door leads to freedom after 2 days of travel. If it is assumed that the prisoner will always select doors 1, 2, and 3 with respective probabilities 0.2 0.5, and 0.3, what is the expected number of days until the prisoner reaches freedom?

(8 Marks)

c) Let X and Y be independent Poisson random variables with respective parameters λ_1 and λ_2. Find the conditional distribution of X given that $X+Y=n$.

(6 Marks)

QUESTION FIVE (20 MARKS)

a) Let $\{X_n, \ n = 1, 2, \ldots\}$ be a sequence of random variables. Briefly explain what is meant the following terms:

   i) X_n converges to X almost surely
   ii) X_n converges to X in probability

(6 Marks)

b) Let $\{X_n, \ n = 1, 2, \ldots\}$ be a sequence of random variables. Show that convergence in quadratic mean implies convergence in probability.

(4 Marks)

c) A research worker wishes to estimate the mean of a population using a sample large enough that the probability will be .95 that the sample mean will not differ from the population mean by more than 25 percent of the standard deviation. How large a sample should he take?

(5 Marks)

d) The amount of impurity in a batch of a chemical product is a random variable with mean value 4.0 g and standard deviation 1.5 g. If 40 batches are independently prepared, what is the probability that the average amount of impurity in these 50 batches is between 3.6 and 4.2 g?

(5 Marks)