LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

FIRST SEMESTER 2025/2026 ACADEMIC YEAR

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (STATISTICS)

STAT 415: STOCHASTIC PROCESSES

STREAM: ***BSc (STATISTICS)*** ***TIME: 2 HRS***

DAY: THURSDAY [8.30 – 10.30 A.M] ***DATE: 05/02/2026***

THIS QUESTION PAPER CONSISTS OF FIVE (5) PAGES

PLEASE DO NOT OPEN UNTIL THE INVIGILATOR SAYS SO.

INSTRUCTIONS:

Answer question **ONE** and any other **TWO** questions.

Mobile phones are not allowed in the examination room.

Candidates are not permitted to write on the examination question paper.

QUESTION ONE (30 MARKS).

(a) A probability generating function with parameter $0 < \alpha < 1$ is given by

$$G(S) = \frac{1 - \alpha(1 - S)}{1 + \alpha(1 - S)}.$$

(i) Find $p_n = P(N = n)$ by expanding the series in powers of S. **(5 Marks)**

(ii) What is the mean of the probability function $\{p_n\}$? **(2 Marks)**

(b) In a casino game based on the standard gambler's ruin problem, the gambler and the dealer each start with 20 tokens, and one token is bet on at each play. The game continues until one player has no further tokens. It is decreed that the probability that the gambler is ruined is 0.52, in order to protect the casino's profit. What should the probability that the gambler wins at each play be? **(4 Marks)**

(c) In an unrestricted random walk with parameters p and q, for what value of p are the mean and variance of the probability distribution of the position of the walk at stage n the same? **(3 Marks)**

(d) A telephone banking service receives an average of 1,000 calls per hour. On average a customer transaction takes one minute. If the calls arrive as a Poisson process, how many operators should the bank employ to avoid an expected accumulation of incoming calls? **(2 Marks)**

(e) An $M(\lambda)/M(\mu)/1$ queue is observed over a long period of time. Regular sampling indicates that the mean length of the queue, including the person being served, is 3, whilst the mean waiting time to completion of service by any customer arriving is 10 minutes. What is the mean service time? **(4 Marks)**

(f) Let $T = [p_{ij}]$ $(i,j = 1,2,3)$ be a transition matrix with entries

$$p_{ij} = \frac{i+j}{6+3i}, \qquad i,j \in \{1,2,3\}$$

(i) Show that T is a row-stochastic matrix. **(4 Marks)**

(ii) What is the probability that a transition between states E_2 and E_3occurs at any step? **(2 Marks)**

(iii) If the initial probability distribution is $P(0) = \left[\frac{1}{2} \ \frac{1}{4} \ \frac{1}{4}\right]$, find the probabilities that states E_1, E_2 and E_3 are occupied after one step (i.e. compute $(1) = P(0)T$). **(4 Marks)**

QUESTION TWO (20 MARKS)

(a) Find the probability that the position of a random walk at the 100^{th} step with probability $p = 0.7$ lies between 40 and 60. **(5 Marks)**

(b) A biased coin is tossed. The probability of a head is p. The coin is tossed until the first head appears. Let the random variable N be the total number of tosses including the first head. Find the

(i) density $P(N = n)$. **(2 Marks)**

(ii) probability generating function $G(s)$. **(4 Marks)**

(iii) expected value of the number of tosses. **(3 Marks)**

(c) In the standard gambler's ruin problem, let the total stake be , the gambler's initial stake be i, and the probability of winning at each play be p. Calculate the probability of ruin if $N = 100, i = 5, p = 0.6$. Also, find the expected duration. **(6 Marks)**

QUESTION THREE (20 MARKS).

(a) Derive the differential-difference equations for the simple death process. **(6 Marks)**

(b) A birth process has a probability generating function

$$G(S,t) = \frac{S}{e^{\lambda t} + S(1-e^{\lambda t})}.$$

(i) What is the initial population size? **(2 Marks)**

(ii) Find the probability that the population size is n at time t. **(6 Marks)**

(iii)Find the mean and variance of the population size at time t. **(6 Marks)**

QUESTION FOUR (20 MARKS)

(a) A bag contains N balls, all green or red. At each stage a ball is taken out of the bag at random and is replaced by a ball of the other colour. Let X_n denote the number of green balls in the bag after n stages.

(i) Explain why $\{X_n: n \geq 0\}$ is a Markov chain with state space

$$S = \{0,1,2,\cdots,N\},$$

and with transition probabilities

$$p_{i,i+1} = \frac{N-i}{N},\ p_{i,i-1} = \frac{i}{N}.$$ **(3 Marks)**

(ii) Show that for this process the stationary distribution $\pi = (\pi_0, \pi_1, \cdots, \pi_N)$ is given by

$$\pi_i = \frac{1}{2^N}\binom{N}{i}, i = 0,1,2,\cdots,N.$$ **(7 Marks)**

(b) A Poisson process with random variable $N(t)$ has probabilities

$$p_n(t) = P[N(t) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}.$$

If $\lambda = 0.5$, calculate the following probabilities associated with the process:

(i) $P[N(3) = 6]$. **(3 Marks)**

(ii) $P[N(3.7) = 4|N(2.1) = 2]$. **(4 Marks)**

(iii)$P[N(7) - N(3) = 3]$. **(3 Marks)**

QUESTION FIVE (20 MARKS).

(a) A queue is observed to have an average length of 2.8 individuals including the one being served. Assuming the usual exponential distributions for both service times and inter-arrival times:

(i) What is the traffic density ρ? **(2 Marks)**

(ii) What is the variance of the queue length? **(2 Marks)**

(b) A branching process starts with two individuals. The offspring probabilities are

$$P_j = \frac{1}{2^{j+1}},\ j = 0,1,2,3,\cdots$$

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Let $G_n(S)$ be the probability generating function of the n^{th}generation. Find the

(i) probability generating function $G_n(s)$. **(5 Marks)**

(ii) probability of extinction by the n^{th}generation. **(3 Marks)**

(iii) probability of ultimate extinction. **(2 Marks)**

(c) Consider the single-server queue with Poisson arrivals occurring with intensity λ, and exponential service times with parameter μ. In the stationary process, the probability P_n that there are n individuals in the queue is given by

$$P_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n, \qquad n = 0,1,2,\ldots.$$

Find its probability generating function, mean and variance of the queue length. **(6 Marks)**