LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

SECOND SEMESTER 2023/2024 ACADEMIC YEAR

FOURTH YEAR EXAMINATION FOR THE DEGREE OF

BACHELOR OF EDUCATION (SCIENCE)/BACHELOR OF EDUCATION (ARTS) AND BACHELOR OF SCIENCE (GENERAL)

MATH 424: MEASURE THEORY

STREAM: R ***TIME: 2 HRS***

DAY:MONDAY [8.30A.M -10.30 A.M] ***DATE: 08/04/2024***

THIS QUESTION PAPER CONSISTS OF FOUR (4) PAGES

PLEASE DO NOT OPEN UNTIL THE INVIGILATOR SAYS SO.

INSTRUCTIONS:
Answers question **ONE** and any other **TWO** questions.

QUESTION ONE (30 MARKS)

a) (i) Define the term partition of a compact interval $[a,b]$. **(1 mark)**

(ii) Let $f:\mathcal{R}\to\mathcal{R}$ be defined by $f(x)=6x-3$ on $[3,6]$. By dividing $[3,6]$ into n equal subintervals, show that $f(x)$ is Riemann integrable on $[3,6]$. **(6 marks)**

b) (i) Explain what is meant by a $\sigma-$algebra on a set X. **(3 marks)**

(ii) Show that a $\sigma-$algebra $\mathfrak{X}$ is closed under countable intersection, that is if $A_i\in\mathfrak{X}$ for $i=1,2,3,\ldots$ then $\bigcap_{i=1}^{\infty}A_i\in\mathfrak{X}$. **(3 marks)**

c) (i) Define a measure on a measurable space $(X,\mathfrak{X})$ **(3 marks)**

(ii) Prove that the measure μ is a monotone set function, that is If $A,B\in\mathfrak{X}$ and $B\subset A$, then $\mu(B)\le\mu(A)$.

(3 marks)

d) (i) Define Lebesgue outer measure μ^* of a set. **(2 marks)**

(ii) Show that for any set A, $\mu^*(A)=\mu^*(A+x)$ where $A+x=\{y+x:y\in A\}$, that is outer measure is translation invariant. **(5 marks)**

e) (i) Define a simple function. **(1 mark)**

(ii) Let φ be a simple function in $M^+(x,\mathfrak{X})$ and $c\ge 0$. Show that $\int c\varphi\,d\mu=c\int\varphi\,d\mu$.

(3 marks)

QUESTION TWO (20 MARKS)

a) Define a cantor set and find its measure. **(8 marks)**

b) Prove that the Dirichlet function $D(x) = \begin{cases} 0, if\ x is an irrational number \\ 1, if\ x is a rational number \end{cases}$

is measurable. **(4 marks)**

c) Prove that the following four statements are equivalent:

$(i) \{x \in X : f(x) < r\} \in x\ \forall r \in \Re$

$(ii) \{x \in X : f(x) \leq r\} \in x \forall r \in \Re$

$(iii) \{x \in X : f(x) > r\} \in x \forall r \in \Re$

$(iv) \{x \in X : f(x) \geq r\} \in x \forall r \in \Re$

(8 marks)

QUESTION THREE (20 MARKS)

a) State Fatou's Lemma. **(3 marks)**

b) Let f_n be a sequence of measurable functions which converge $\mu - a.e$ in X to a function f. Suppose there exists a function g which is integrable on A such that $|f_n(x)| \leq g(x) \forall x$. Then prove that: $\lim_{n \to \infty} \int f_n\, d\mu = \int f\, d\mu$ **(10 marks)**

c) Let $f(x)$ and $g(x)$ be two functions such that $f, g : X \to \Re \in \mathfrak{X}$. then prove that:

(i) $f^2 \in \mathfrak{X}$ **(2 marks)**

(ii) $(f + g) \in \mathfrak{X}$ **(2 marks)**

(iii) $(fg) \in \mathfrak{X}$ **(2 marks)**

d) Let $(X, \mathfrak{X}, \boldsymbol{\mu})$ be a measure space and f be a measurable function. When do we say that f is integrable? **(1 mark)**

QUESTION FOUR (20 MARKS)

a) When is a subset E of $\Re$ said to be Lebesgue non- measurable?. Hence or otherwise show that if E is L-non-measurable, then it is possible to find a proper subset A of E such that $\mu^*(A) > 0$. **(5 marks)**

b) Prove that if $E_1, E_2 \in \mathfrak{M}$ (where $\mathfrak{M}$ the class of Lebesgue measurable sets), then $E_1 \cap E_2 \in \mathfrak{M}$ **(10 marks)**

c) Show that if $\mu^*(E) = 0$ then E is L- measurable. **(3 marks)**

d) If I is an interval in $\Re$, show that I is uncountable. **(2 marks)**

.

QUESTION FIVE (20 MARKS)

a) Let f_n be a non-decreasing sequence of non-negative measurable functions with limit f. Then show that:

$$\int f\, d\mu = \lim_{n\to\infty}\int f_n d\mu.$$ **(10 marks)**

b) Prove that if a function f is Riemann integrable on $[a,b]$ then it is also Lebesgue integrable on $[a,b]$ and

$$(L)\int_a^b f(x)\,dx = (R)\int_a^b f(x)\,dx$$

where $(R)\int_a^b f(x)\,dx$ denotes the Riemann integral and $(L)\int_a^b f(x)\,dx$ denotes the Lebesgue integral. **(10 marks)**