LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

SECOND SEMESTER 2023/2024 ACADEMIC YEAR

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (STATISTICS)

STAT 425: BAYESIAN INFERENCE

STREAM: R ***TIME: 2 HRS***

DAY: TUESDAY[8.30AM - 10.30A.M] ***DATE: 16/04/2024***

THIS QUESTION PAPER CONSISTS OF FIVE (5) PAGES

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INSTRUCTIONS: Answer **QUESTION ONE** and any other **TWO** questions.

QUESTION ONE (30 MARKS)

a) Suppose that the prior distribution of some parameter θ is a gamma distribution for which the mean is 6 and the variance is 12. Determine the prior probability density function of θ.

(**5 Marks**)

b) The heights of adult females are approximately normally distributed with unknown mean θ cm and a standard deviation of 8 cm. Assume also that the prior distribution of θ is normal with mean 140 cm and a standard deviation of 10 cm. Suppose that a random sample of size $n =$ 20 adult females selected from this population yielded average height of 157.5 cm. Find the probability that the population mean height is between 155 and 159. **(6 Marks)**

c) Let $\underline{X}$ be a random vector of observations from a distribution with density function $f(x|\theta)$, where $\theta \in \Theta$. Let $\pi(\theta)$ be the prior density of θ. If the value of θ is to be estimated by using the absolute error loss function, show that the Bayes estimator $\delta^*(\underline{X})$ of θ is the median of the posterior distribution $\pi(\theta|\underline{x})$. **(6 Marks)**

d) Suppose that the proportion θ of defective items in a large shipment is unknown, and the prior distribution of θ is the beta distribution for which the parameters are $\alpha = 5$ and $\beta = 10$. Suppose also that 20 items are selected at random from the shipment, and that exactly six of these items are found to be defective.

i) Find the posterior density function of θ.

ii) If the squared error loss function is used, what is the Bayes estimate of θ? **(6 Marks)**

e) Suppose that the scores on an examination are normally distributed with unknown mean θ and standard deviation 15. Assume a normal prior distribution for θ with a mean of 80 minutes and a standard deviation of 12 minute. If a random sample of 20 students has an average score of 87, find a 95% HPD credible interval for θ.

(6 Marks)

QUESTION TWO (20 MARKS)

a) Assume that the prior distribution for the proportion θ of dinks from a vending machine that overflow is as follows: $\pi(0.08) = 0.6, \pi(0.1) = 0.3$ and $\pi(0.15) = 0.1$. If 3 of the next 12 drinks from the machine overflow, find the posterior probability distribution of θ. **(6 Marks)**

b) Suppose that $X_1, X_2, \cdots, X_n$ are IID with common density function $f(x;\theta) = \begin{cases} \theta e^{-\theta x} & x > 0 \\ 0 & otherwise \end{cases}$,

where $\theta(>0)$ is the unknown parameter. Assume the prior density of θ is

$\pi(\theta) = \begin{cases} \frac{1}{16}\theta^2 e^{-\frac{\theta}{2}} & \theta > 0 \\ 0 & elsewhere \end{cases}$. Derive the posterior density function of θ given that $T = t$ where

$T = \sum_{i=1}^{n} X_i$. **(8 Marks)**

c) Suppose that the proportion θ of in large lot is unknown and has the following prior density function $\pi(\theta) = 12\theta(1-\theta)^2$, $0 < \theta < 1$. Suppose that a random sample of 10 apples is drawn from the lot and it is found that 3 are bad. Find the Bayes estimate of θ with respect to the zero-one error loss function. **(6 Marks)**

QUESTION THREE (20 MARKS)

Suppose that we have the random variables $X_1, X_2, \cdots, X_n$ which are IID Poisson (θ), where $\theta > 0$ is the unknown parameter. Suppose that the prior distribution of θ is $Gamma(\alpha, \beta)$, where $\alpha > 0$ and $\beta > 0$ are known.

a) Derive the posterior density of θ given $T = t$ where $T = \sum X_i$ **(9 Marks)**

b) If the value of θ is to be estimated by using the squared error loss function, what is the Bayes estimator $\delta^*(T)$ of θ? **(4 Marks)**

c) Show that the Bayes estimate $\delta^{*}(T)$ obtained in part b) is a weighted average of the form $\gamma_n \overline{X} + (1-\gamma_n)\frac{\alpha}{\beta}$ and that $\gamma_n \to 1$ as $n \to \infty$. **(4 Marks)**

d) Is $\{\pi(\theta): \theta > 0\}$ a conjugate family of prior distributions for the Poisson(θ) family of distributions? Explain your answer. **(3 Marks)**

QUESTION FOUR (20 MARKS)

a) Suppose that two random variables θ and τ have the joint normal-gamma distribution with hyperparameters $\mu_0 = 4, \lambda_0 = 0.5, \alpha_0 = 2,$ and $\beta_0 = 8$. Find the value of $P(5.6094 < \theta < 14.5981)$. **(5 Marks)**

b) Suppose that in a random sample of size $n = 10$ drawn from the normal distribution with unknown mean θ and unknown precision τ it is found that $\sum_{i=1}^{10} x_i = 28$ and $\sum_{i=1}^{n} x_i^2 = 86.4$.

Suppose also that the joint prior distribution of θ and τ is the normal-gamma distribution specified in part (a).

i) Determine the posterior hyperparameters $\mu_1, \lambda_1, \alpha_1$ and β_1. **(8 Marks)**

ii) Find the shortest possible interval such that the posterior probability that θ lies in the interval is 0.99. **(5 Marks)**

iii) Argue that the above interval is a 99% HPD credible interval for θ. **(2 Marks)**

QUESTION FIVE (20 MARKS)

a) Suppose also that the joint prior distribution of θ and τ is the normal-gamma distribution such that $E(\theta) = 5$, $Var(\theta) = 4$, $E(\tau) = \frac{1}{4}$ and $Var(\tau) = \frac{1}{32}$. Find the prior hyperparameters $\mu_0, \lambda_0, \alpha_0$ and β_0. **(6 Marks)**

b) Suppose that the scores on a college placement test in mathematics follow a normal distribution with mean 72 and unknown variance o^2. Assume that the prior distribution of σ^2 is the inverted gamma distribution $IG(3,4)$. If a random sample of $n = 20$ students yielded $\sum_{i=1}^{20} x_i = 1365, \sum_{i=1}^{20} x_i^2 = 93984$, find the probability that the population variance is between 36.5369 and 64.3072. (**6 Marks**)

c) Let $X_1, X_2, \cdots, X_{10}$ be a random sample from $N[\theta, 5]$ distribution, where $\theta \in \Re$ is the unknown parameter. Assume that the prior distribution of θ is $N[5,2]$. Suppose that the sample produced $\sum_{i=1}^{10} x_i = 54.2$. Use the Bayes test procedure to test $H_0 : \theta \leq 5.2$ versus $H_1 : \theta > 5.2$. Assume that the loss from choosing an incorrect decision is 1 unit, and the loss from choosing a correct decision is 0. (**8 Marks**)