LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

1ST SEMESTER 2022/2023 ACADEMIC YEAR

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF EDUCATION (SCIENCE)

MATH 422: FUNCTIONAL ANALYSIS

STREAM: R ***TIME: 2 HRS***

DAY: MONDAY [8.30AM – 10.30AM] ***DATE: 05/12/2022***

THIS QUESTION PAPER CONSISTS OF THREE (3) PAGES

PLEASE DO NOT OPEN UNTIL THE INVIGILATOR SAYS SO.

INSTRUCTIONS: - Answer question ONE and any other TWO questions

QUESTION ONE (30 MARKS)

(a) Define what is meant by
  i) A vector space over $\Re$
  ii) A metric space (X, d) (6 marks)

(b) Let (X, d) be a metric space
  i) Give the definition of a bounded subset A of X
  ii) Prove that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$ (6 marks)

(c) i) Give the definition of a compact subset A of a metric space X.

  ii) Show that the set (1, 2) is not compact in $\Re$. (6 marks)

(d) State Cantor's intersection theorem. (4 marks)

(e) Let (X, d) be a metric space
  i) Explain the meaning of a subset A being dense in X
  (3 marks)
  ii) State a dense and a nowhere dense subset in $\Re$
  (2 marks)
  iii) Give the reasons for your answer in ii)
  (3 marks)

QUESTION TWO (20 MARKS)

(a) Define
  i) A convergent sequence (x_n)
  ii) A Cauchy sequence, (x_n), on a metric space (X, d) (6 marks)

(b) i) Prove that every convergent sequence in a metric space is a Cauchy sequence
  (4 marks)
  ii) Explain the meaning of a *complete metric space* (3 marks)

(c) i) Show that A= $(0, 1) \subseteq \Re$, with the standard metric d on $\Re$, is not a complete metric space. (4 marks)

  ii) State the theorem which explains why $[0, 1] \subseteq \Re$, is complete (3 marks)

QUESTION THREE (20 MARKS)

(a) Given a compact subset A of a metric space (X, d), prove that
  i) A is bounded in X.
  ii) A is closed

(10 marks)

(b) Let X and Y be metric spaces and f: (X, d) →(Y, ρ) be a function from X to Y.

i) Define what is meant by, f is continuous at a point $x_0 \in X$.

(5 marks)

ii) .Given the function, f: ($\Re$, d) →($\Re$, d) defined by $f(x)= x^2-2x$ where d is the standard metric in $\Re$, Show that f is continuous at $x=1$.

(5 marks)

QUESTION FOUR (20 MARKS)

(a) Let f: (X, d) →(Y, ρ) be a continuous function from a metric space X to Y.

(i) Show that if A is a compact subset of X, f(A) is a compact subset of Y.

(10 marks)

(ii) Prove that f^{-1}(A) is open whenever A is an open subset of Y.

(6 marks)

(b) Given X={1, 2, 3, 4} and τ={ϕ, {2}, {2, 3}, {1, 2}, X}, determine whether (τ, X) is a topological space or not.

(4 marks)

QUESTION FIVE (20 MARKS)

(a) Let (X, d) be a metric space and A, B be subsets of X. Show that

i) $A^{\circ}=(\overline{A^c})^c$ (5 marks)

ii) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$ (5 marks)

(b) State Baire's category theorem.
Use the theorem to show that the set of irrationals is of second category.

(5 marks)

(c) Define the diameter of a subset A in a metric space (X, d). Hence show that diam (A)=diam ($\overline{A}$)

(5 marks)