LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

FIRST SEMESTER 2025/2026 ACADEMIC YEAR

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (GENERAL)

MATH 422: FUNCTIONAL ANALYSIS

STREAM: R ***TIME: 2 HRS***

DAY: FRIDAY [11.30 – 13.30 P.M] ***DATE: 30/01/2026***

THIS QUESTION PAPER CONSISTS OF THREE (3) PAGES

PLEASE DO NOT OPEN UNTIL THE INVIGILATOR SAYS SO.

INSTRUCTIONS:

Answer question ONE and any other TWO questions

QUESTION ONE (30 MARKS)

a) Let (X, d) be a metric space, where $X = \Re$ with the standard metric d on $\Re$

i) Show that $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z in X **(4 Marks)**

ii) Prove that the map defined by;

$f(x) = \frac{1}{3}x^2 + \frac{2}{3}$, , is a contraction on $A = [0, 1.4]$ **(4 Marks)**

b) The set $A = \{(x, y) : (x-1)^2 + y^2 < 4\}$ is a subset of $\Re^2$, with $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$

i) Determine whether A is bounded or not **(4 Marks)**

ii) Verify that d is a metric on A **(4 Marks)**

c) i) Explain the meaning of an open subset A of a metric space X. **(4 Marks)**

ii) If A and B are open subsets of a metric space X, show that $(A' \cap B')$ is closed. **(4 Marks)**

d) Let (X, d) be a metric space

i) What is the meaning of a subset A *being dense in* X? **(3 Marks)**

ii) State a dense and a nowhere dense subset in $\Re$ **(3 Marks)**

QUESTION TWO (20 MARKS)

a) Prove that a convergent sequence (x_n) in a metric space X has a unique limit **(6 Marks)**

b) i) Show that every convergent sequence in a metric space is a Cauchy sequence **(4 Marks)**

ii) Using a counterexample, illustrate that a Cauchy sequence need not converge in a general metric space **(3 Marks)**

c i) Show that $A = (0, 1) \subseteq \Re$, with the standard metric d on $\Re$, is not a complete metric space. **(4 Marks)**

ii) Explain why $[0, 1] \subseteq \Re$, is complete. **(3 Marks)**

QUESTION THREE (20 MARKS)

a) Given a compact subset A of a metric space (X, d), prove that

i) A is bounded in X.

ii) A is closed **(10 Marks)**

b) Let X and Y be metric spaces and $f: (X, d) \to (Y, \rho)$ be a function from X to Y.

i) Define what is meant by, f is continuous at a point $x_0 \in X$. **(5 Marks)**

ii) .Given the function, $f: (\Re, d) \to (\Re, d)$ defined by $f(x) = x^2 - 3x + 2$ where d is the standard metric in $\Re$. Show that f is continuous at $x=1$. **(5 Marks)**

QUESTION FOUR (20 MARKS)

a) Let $f: (X, d) \to (Y, \rho)$ be a contraction map from a metric space X to Y.

i) Show that f is continuous **(5 Marks)**

ii) Verify Banach's fixed theorem using the map $f(x) = \frac{1}{4}x^2 + \frac{3}{4}$ on $A=[0, 1]$ with the standard metric d on $\Re$. **(5 Marks)**

b) Determine whether A is dense in X or not

i) $A=\{2, 3, 4\} \subset X$, where $X=\{1, 2, 3, 4\}$ and $\tau=\{\phi, \{2\}, \{2, 3\}, \{1, 2\}, \{1, 2, 3\}, X\}$, a topology on X

(5 Marks)

ii) $A=(1, 2]$, $X=\Re$ **(5 Marks)**

QUESTION FIVE (20 MARKS)

a) Let (X, d) be a metric space and A, B be subsets of X. Show that

i) $A^\circ = (\overline{A^c})^c$ **(5 Marks)**

ii) $(\overline{A \cup B}) = \overline{A} \cup \overline{B}$ **(5 Marks)**

b) Use Baire's category theorem to show that the set of irrationals is of second category.

(5 Marks)

c) Let $f: (X, d) \to (Y, \rho)$ be a continuous function between two metric spaces. Show that $f^{-1}(A)$ is open whenever A is an open subset of Y. **(5 Marks)**