LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

FIRST SEMESTER 2025/2026 ACADEMIC YEAR

THIRD YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (STATISTICS)

STAT 311: MATHEMATICAL STATISTICS III

STREAM: R ***TIME: 2 HRS***

DAY: FRIDAY [8.30 – 10.30 A.M] ***DATE: 06/02/2026***

THIS QUESTION PAPER CONSISTS OF FOUR (4) PAGES

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INSTRUCTIONS:

Answer question **ONE** and any other **TWO** questions.

Mobile phones are not allowed in the examination room.

Candidates are not permitted to write on the examination question paper.

QUESTION ONE (30 MARKS)

(a) Let X be a random vector. Show that

$$Var\ (A\boldsymbol{X} + \boldsymbol{b}) = A\ Var(\boldsymbol{X})A^T.$$

where $\boldsymbol{A}$ is a matrix of constants and $\boldsymbol{b}$ is a vector of constants. **(4 Marks)**

(b) Let the random vector $\boldsymbol{V} \sim N_4(\boldsymbol{\mu},\ \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix},\quad \boldsymbol{\Sigma} = \begin{pmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -3 & 7 \end{pmatrix}.$$

Assume that $\boldsymbol{V}$ is partitioned as $V = (y_1, y_2, x_1\ x_2)^T$. Calculate the conditional

(i) Expectation $E(\boldsymbol{y}|\boldsymbol{X})$. **(6 Marks)**

(ii) Covariance matrix $Cov(\boldsymbol{y}|\boldsymbol{X})$. **(4 Marks)**

(c) Let $(X_1, \cdots, X_n)$ have a multinomial distribution with m trials and cell probabilities $P_1, \cdots, P_n$. Given that $X_i \sim Binomial\ (m,\ p_i)$, $X_j \sim Binomial\ (m,\ p_j)$, and $X_{i+j} \sim Binomial\ (m, p_i + p_j)$. Use this information to show that for every i and j, $Cov\ (X_i, X_j) = -mp_ip_j$. **(4 Marks)**

(d) Give the statement for the Central Limit Theorem (CLT). **(3 Marks)**

(e) A random variable has the probability distribution function

$$f(x) = \frac{1}{2^{x+1}},\ x = 0,1,2,3,\cdots$$

Find

(i) the probability generating function $G_X(S)$. **(4 Marks)**

(ii) use the p.g.f. to find the variance. **(5 Marks)**

QUESTION TWO (20 MARKS)

(a) Consider the following probability density function: $f(x_1, x_2, x_3) = \frac{2}{3}\,(x_1 + x_2 + x_3)$ where $0 < x_1, x_2, x_3 < 1$. Calculate the

(i) joint distribution for X_1 and X_2. **(3 Marks)**

(ii) marginal distribution for X_3. **(3 Marks)**

(iii) conditional distribution $f(x_1, x_3|x_2)$. **(3 Marks)**

(iv) conditional expectation $E(X_1X_3|X_2 = x_2)$. **(5 Marks)**

(b) Suppose that the random vector $\boldsymbol{X}$ is normally distributed with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, and $Y = a^T\boldsymbol{X}$, $Z = b^T\boldsymbol{X}$ are linear functions of $\boldsymbol{X}$. Prove that Y and Z are independent if $a^T\boldsymbol{\Sigma}\,\mathrm{b} = 0$. **(6 Marks)**

QUESTION THREE (20 MARKS)

(a) A random variable X has the probability generating function $G_X(S)$ where

$$G_X(S) = 0.4096 + 0.4096S + 0.1528S^2 + 0.0256S^3 + 0.0016S^4.$$

(i) Determine the probability $P(X = 3)$ using $G_X(S)$. **(3 Marks)**

(ii) Given that X follows a binomial distribution. Determine the parameter values of the distribution of X. **(3 Marks)**

(b) Let Y have the distribution of the geometric form (modified geometric or decapitated geometric) given by $P(Y = k) = q^{k-1}p,\ k = 1,2,3,\cdots$ where $q = 1 - p$. Find the

(i) Probability generating function (p.g.f.) of Y. **(3 Marks)**

(ii) Expected value $E(Y)$ and the variance $Var\,(Y)$ using the probability generating function derived in part (i). **(6 Marks)**

(c) A probability generating function with parameter $0 < \alpha < 1$ is given by

$$G_X(S) = \frac{1 - \alpha(1 - S)}{1 + \alpha(1 - S)}.$$

Find $p_x = P(X = x)$ by expanding the series in powers of S. **(5 Marks)**

QUESTIONFOUR (20 MARKS)

Suppose that we have a multivariate normal variable $\boldsymbol{y} \sim N_3(\boldsymbol{\mu},\ \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix},\ \ \boldsymbol{\Sigma} = \begin{pmatrix} 6 & 1 & -2 \\ 1 & 13 & 4 \\ -2 & 4 & 4 \end{pmatrix}.$$

Find the

(i) distribution of $z = 2y_1 - y_2 + 3\,y_3$. **(3 Marks)**
(ii) joint distribution of $z_1 = y_1 + y_2 + y_3$, and $z_2 = y_1 - y_2 + 2\,y_3$. **(3 Marks)**
(iii) marginal distributions of y_1 and y_2. **(2 Marks)**
(iv) joint distribution of y_1 and y_2. Are y_1 and y_2 independent? **(3 Marks)**
(v) conditional expectation $E[Y_1|Y_2]$. **(2 Marks)**
(vi) conditional variance $Var\,[y_1|y_2]$. **(2 Marks)**
(vii) partial correlation coefficient $\rho_{12.3}$. **(5 Marks)**

QUESTION FIVE (20 MARKS)

(a) Explain briefly the following convergence concepts.

(i) Convergence in probability. **(2 Marks)**

(ii) Almost sure convergence. **(2 Marks)**

(iii) Convergence in distribution. **(2 Marks)**

(b) Let $X_i,\ i = 1,2,\cdots$ be independent $Bernoulli\ (p)$ random variables and let

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Show that

(i) $\sqrt{n}(Y_n - p)$ converges in distribution to $N[0,\ p(1-p)]$. **(4 Marks)**

(ii) For $p \neq 1/2$, the estimate of variance $Y_n(1 - Y_n)$ converges in distribution as follows $\sqrt{n}[Y_n(1-Y_n) - p(1-p)] \longrightarrow N[0,(1-2p)^2 p(1-p)]$. **(5 Marks)**

(c) Suppose $X_1, \cdots, X_n$ are a random sample from a negative binomial distribution $NB(r, p)$. The Central Limit Theorem (CLT) tells us that

$$\sqrt{n}\left[\overline{X} - \frac{r(1-p)}{p}\right]\frac{p}{\sqrt{r(1-p)}} \sim N(0,1).$$

Suppose that $r = 20, p = 1/2$, and $n = 20$, use the CLT to approximate the probability $P(\overline{X} \leq 11)$. **(5 Marks)**