LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

FIRST SEMESTER 2025/2026 ACADEMIC YEAR

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (STATISTICS)

STAT 412: INTRODUCTION TO MEASURE AND PROBABILITY

STREAM: R ***TIME: 2 HRS***

DAY: TUESDAY [8.30 – 10.30 A.M] ***DATE: 03/02/2026***

THIS QUESTION PAPER CONSISTS OF THREE (3) PAGES

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INSTRUCTIONS: Answer question **ONE** and any other **TWO** questions

QUESTION ONE (30 MARKS)

a) Let $\Omega = \{a,b,c\}$. Consider the classes $\mathcal{F}_1 = \{\{a\},\{b,c\},\Omega,\phi\}$ and $\mathcal{F}_2 = \{\{c\},\{a,b\},\Omega,\phi\}$. Verify that $\mathcal{F}_1$ and $\mathcal{F}_2$ are both algebras but $\mathcal{F}_1 \cup \mathcal{F}_2$ is not an algebra. (**4 Marks**)

b) Let μ^* be the Lebesgue outer measure on $\mathbb{R}$. Prove that $A \cup B$ is Lebesgue measurable whenever A and B are. (**5 Marks**)

c) Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $B \epsilon \mathfrak{R}$ satisfies $\mathbb{P}(B) > 0$. Show that $\mathbb{Q}: \mathcal{F} \to [0,1]$, where $\mathbb{Q}(A) = \mathbb{P}(A/B)\ \forall A \in \mathcal{F}$, is a probability measure on Ω. If $C \in \mathcal{F}$ and $\mathbb{Q}(C) > 0$, show that $\mathbb{Q}(A/C) = \mathbb{P}(A/B \cap C)$. (**7 Marks**)

d) Let X be a Poisson random variable with parameter λ. Show that $\mathbb{P}[X\ is\ odd] = \frac{1}{2}\left(1 - e^{-2\lambda}\right)$ (**4 Marks**)

e) Let $X_1, X_2, \ldots, X_n$ be IID exponential random variables with density function $f(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}$, $x > 0$. Find the characteristic function of the random variable $Y = \frac{2}{\lambda}\sum_{j=1}^{n} X_j$ and identify its distribution. (**6 Marks**)

f) The amount of impurity in a batch of a chemical product is a random variable with mean value 3.6 g and standard deviation 1.2 g. If 40 batches are independently prepared, what is the probability that the average amount of impurity in these 50 batches is between 3.5 and 4.1 g? (**4 Marks**)

QUESTION TWO (20 MARKS)

a) Let $\mathcal{F}$ and $\mathcal{G}$ be σ-algebra of subsets of Ω. Let $\mathcal{H} = \mathcal{F} \cap \mathcal{G}$ be the collection of subsets of Ω lying in both $\mathcal{F}$ and $\mathcal{G}$. Show that $\mathcal{H}$ is σ-algebra. (**6 Marks**)

b) Let μ be a measure on an algebra $\mathscr{F}$, and let $A_1, A_2, \ldots, A_k \in \mathcal{F}$, $1 \le k < \infty$. Prove that $\left(\bigcup_{i=1}^{k} A_i\right) \le \sum_{i=1}^{k} \mu(A_i)$. (**9 Marks**)

c) Let $(\Omega, \mathcal{F}, \mu)$ μ^* be a measure space, and let $E \in \mathcal{F}$. Define $\mu_E: \mathcal{F} \longmapsto [0, +\infty]$ by $\mu_E(A) = \mu(A \cap E)$, $A \in \mathcal{F}$. Prove that μ_E is a measure (**5 Marks**)

QUESTION THREE (20 MARKS)

a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Prove that $|\mathbb{P}(A) - \mathbb{P}(B)| \le \mathbb{P}(A \backslash B) + \mathbb{P}(B \backslash A)$, for all $A, B \in \mathcal{F}$. (**3 Marks**)

b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that $\mathbb{P}$ is finitely subadditive, that is, for any $\{A_i\}_{i=1}^{n} \subset \mathcal{F}$ where $n \in \mathbb{N}$; one has that $\mathbb{P}(\bigcup_{i=1}^{n} A_i) \le \sum_{i=1}^{n} \mathbb{P}(A_i)$ (**5 Marks**)

c) Let $A_1, A_2, \ldots, A_n$ be events in the sample space Ω. If $A_1, A_2, \ldots, A_n$ are independent and $\mathbb{P}(A_i) = \theta, i = 1,2, \ldots, n$, find an expression for $\mathbb{P}(\bigcup_{i=1}^{n} A_i)$ (**4 Marks**)

d) A laboratory blood test is 95 % effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 5% of the healthy persons tested. If 0.8 % of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive? (**8 Marks**)

QUESTION FOUR (15 MARKS)

a) Show that if X is a Poisson random variable with parameter λ, then $E\left[X^n\right] = \lambda E\left[X+1\right]^{n-1}$. Use this result to compute $E\left[X^3\right]$. (**6 Marks**)

b) A prisoner is trapped in a cell containing 3 doors. The first door leads to a tunnel that returns him to his cell after 2 days' travel. The second leads to a tunnel that returns him to his cell after 4 days' travel. The third door leads to freedom after 1 day of travel. If it is assumed that the prisoner will always select doors 1, 2, and 3 with respective probabilities .5, .3, and .2, what is the expected number of days until the prisoner reaches freedom? (**8 Marks**)

c) Let X and Y be independent Poisson random variables with respective parameters λ_1 and λ_2. Find the conditional distribution of X given that $X + Y = n$. (**6 Marks**)

QUESTION FIVE (20 MARKS)

a) Let $\{X_n, \; n = 1, 2, \dots\}$ be a sequence of random variables. Briefly explain what is meant by X_n converges to X almost surely. (**4 Marks**)

b) Let $X_1, X_2, \cdots,$ be a sequence of random variables, with $E(X_n) = 8$ and $Var(X_n) = \frac{1}{\sqrt{n}}$ for each n.

Prove that $\{X_n\}_{n\geq 1}$ converges to 8 in probability (**6 Marks**)

c) Suppose that some distribution with an unknown mean has variance equal to 1. How large a sample must be taken in order that the probability wi1l be at least .95 that the sample mean $\overline{X}_n$ will lie within 0.5 of the population mean. (**5 Marks**)

d) A coin is weighted so that its probability of landing on heads is 10%. Suppose the coin is flipped 200 times. Use Chebyshev's inequality to determine a bound for the probability that it lands on heads at least 120 times. (**5 Marks**)